

# Polynomial extensions of the Weyl $C^*$ -algebra

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## Abstract

We introduce higher order (polynomial) extensions of the unique (up to isomorphisms) non trivial central extension of the Heisenberg algebra. Using the boson representation of the latter, we construct the corresponding polynomial analogue of the Weyl  $C^*$ -algebra and use this result to deduce the explicit form of the composition law of the associated generalization of the 1-dimensional Heisenberg group. These results are used to calculate the vacuum characteristic functions as well as the moments of the observables in the Galilei algebra. The continuous extensions of these objects gives a new type of *second quantization* which even in the quadratic case is quite different from the quadratic Fock functor studied in [7].

## 1 Introduction

The program to develop a theory of renormalized higher ( $\geq 2$ ) powers of white noise [6] has led to investigate the connections between central extensions and renormalization. This has, in its turn, led to the discovery of the unique (up to isomorphisms) non trivial central extension of the 1-dimensional Heisenberg algebra  $heis(1)$  and of its identification with the Galilei algebra (cf [1], [5]). In the present paper we will work in the 1-dimensional Schroedinger representation where  $heis(1)$  can be realized as the real linear span of  $\{1, p, q\}$  (central element is the identity 1) and the operators

$$(q\varphi)(x) := x\varphi(x); \quad p\varphi(x) := \frac{1}{i} \frac{\partial}{\partial x} \varphi(x) \quad ; \quad \varphi \in L^2(\mathbb{R})$$

are defined on suitable domains in  $L^2(\mathbb{R})$  and satisfy the Heisenberg commutation relations

$$[q, p] = i$$

and all other commutators are zero. In the same representation  $Ceheis(1)$  can be realized as the real linear span of

$$\{1, p, q, q^2\}$$

It has been recently proved that the current algebra over  $\mathbb{R}$  of this algebra admits a non-trivial (suitably defined) Fock representation: this is equivalent to prove the infinite divisibility of the vacuum distributions of the operators of the form

$$\alpha q^2 + \beta q + \gamma p \quad ; \quad \alpha, \beta, \gamma \in \mathbb{R}$$

For each  $N \in \mathbb{N}$ , the real Lie algebra with generators

$$\{1, p, q, \dots, q^N\}$$

in the following denoted  $heis(1, 1, N)$  is a natural generalization of the Galilei algebra. The analogy with the usual Heisenberg algebra, i.e.  $heis(1, 1, 1)$ , naturally suggests the following problems:

- 1) To describe the Lie group generated in the Schroedinger representation by  $heis(1, 1, N)$ .
- 2) To find an analogue of the Weyl commutation relations for the group in item 1). This is equivalent to describe the  $(1, N)$ -polynomial extensions of the Heisenberg group.
- 3) To determine the vacuum distribution of the nontrivial elements of  $heis(1, 1, N)$  and their moments. These elements have the form

$$up + \sum_{j=0}^N a_j q^j$$

with  $u \neq 0$  and some  $a_j \neq 0$  with  $j \geq 2$ .

- 4) To extend the constructions of items 1), 2) above to the continuous case, i.e. to describe the current  $*$ -Lie algebra of  $heis(1, 1, N)$  over  $\mathbb{R}$ .

The solutions of these problems are discussed in the following. In particular we show that the group structure emerging from problem 2) is non trivial and yet controllable (see Theorem 1) thus allowing a solution of problem 3) (see Proposition 2). Problem 4) was solved in [2] for the Galilei algebra ( $N = 2$ ) but the technique used there cannot be applied to the polynomial case. In section 4 we introduce the *q-projection-method* in order to overcome this difficulty. This reduces the problem to the calculation of the expectation value of functionals of the form  $\exp(P(X) + icX)$  where  $P$  is a polynomial with real coefficients,  $c$  is a real number, and  $X$  is a standard Gaussian random variable. If  $P$  has degree 2, this gives a different proof of result obtained in [1]. The calculation of the moment in subsection 4.3 and the continuous second quantization versions of the polynomial extensions of the Weyl  $C^*$ -algebra are new even in the quadratic case.

## 2 Polynomial extensions of the Heisenberg algebra: Discrete case

The one-mode Heisenberg algebra is the  $*$ -Lie algebra generated by  $b, b^+, 1$  with commutation relations

$$[b, b^+] = 1, [b, 1] = [b^+, 1] = 0$$

and involution

$$(b)^* = b^+$$

The associated position and momentum operators are defined respectively by

$$q = \frac{b^+ + b}{\sqrt{2}}, \quad p = \frac{b - b^+}{i\sqrt{2}}$$

We will use the known fact that, if  $\mathcal{L}$  is a Lie algebra and  $a, b \in \mathcal{L}$  are such that

$$[a, [a, b]] = 0$$

then, for all  $w, z \in \mathbb{C}$

$$e^{za} e^{wb} e^{-za} = e^{w(b+z[a,b])} \tag{1}$$

**Proposition 1** For all  $u, w \in \mathbb{C}$  ( $w \neq 0$ ) and a polynomial  $P$  in one determinate, we have

$$e^{iwp+iuP'(q)} = e^{iu \frac{P(q+w)-P(q)}{w}} e^{iwp} \quad (2)$$

**Proof.** Let  $P$  a polynomial in one determinate and  $w, z \in \mathbb{C}$  such that  $w \neq 0$ . Then from the identity

$$[P(q), p] = iP'(q)$$

It follows that

$$[P(q), [P(q), p]] = 0$$

Hence (1) implies that

$$e^{izP(q)} e^{iwp} e^{-izP(q)} = e^{iw(p-zP'(q))} \quad (3)$$

But, one has also

$$e^{iwp} e^{-izP(q)} = (e^{iwp} e^{-izP(q)} e^{-iwp}) e^{iwp} = e^{-izP(q+w)} e^{iwp} \quad (4)$$

The identities (3) and (4) imply

$$e^{iw(p-zP'(q))} = e^{izP(q)} e^{-izP(q+w)} e^{iwp} \quad (5)$$

With the change of variable  $u := -wz$ , (5) becomes (2).  $\square$

Since the real vector space of polynomials of degree  $\leq N$  with coefficients in  $\mathbb{R}$ , denoted in the following

$$\mathbb{R}_N[x] := \{P' \in \mathbb{R}[x] : \text{degree}(P') \leq N\}$$

has dimension  $N + 1$ , in the following we will use the identification

$$P'(x) = \sum_{k=0}^N a_k x^k \equiv (a_0, a_1, \dots, a_N) \in \mathbb{R}^{N+1}$$

Now, for all  $w \in \mathbb{R}$ , we define the linear map  $T_w$  by  $T_0 := id$  if  $w = 0$  and if  $w \neq 0$

$$T_w : P' \in \mathbb{R}_N[x] \rightarrow \frac{1}{w} \int_0^w P'(\cdot + y) dy \in \mathbb{R}_N[x] \quad (6)$$

so that

$$T_w P' = \frac{P(\cdot + w) - P}{w} \quad (7)$$

where  $P$  is any primitive of  $P'$ .

## 2.1 Properties of the maps $T_w$

**Lemma 1** *For each  $w \in \mathbb{R} \setminus \{0\}$  the linear map  $T_w$  is invertible. Moreover, we have*

$$T_w^{-1} = \sum_{k=0}^N (-1)^k (T(w))^k$$

where  $T(w)$  is a lower triangular nilpotent matrix whose coefficients, with respect to the monomial real basis  $\{1, x, \dots, x^N\}$  of  $R_N[x]$ , are given by

$$t_{mn}(w) = \chi_{\{m < n\}} \frac{n!}{(n+1-m)!m!} w^{n-m}, \quad 0 \leq m, n \leq N \quad (8)$$

where for all set  $I$

$$\chi_{\{m < n\}} = \begin{cases} 0, & \text{if } m \geq n \\ 1, & \text{if } m < n \end{cases}$$

**Proof.** Let  $k \in \{0, 1, \dots, N\}$ . If we take

$$P'(x) := x^k$$

then, in (7) we can choose

$$P(x) = \frac{x^{k+1}}{k+1}$$

so that

$$\frac{P(x+w) - P(x)}{w} = \frac{1}{w(k+1)} [(x+w)^{k+1} - x^{k+1}]$$

Since

$$\begin{aligned} (x+w)^{k+1} &= \sum_{h=0}^{k+1} \binom{k+1}{h} x^h w^{k+1-h} \\ &= x^{k+1} + \sum_{h=0}^k \binom{k+1}{h} x^h w^{k+1-h} \end{aligned}$$

it follows that

$$(x+w)^{k+1} - x^{k+1} = \sum_{h=0}^k \binom{k+1}{h} x^h w^{k+1-h}$$

Hence

$$T_w 1 = 1 \quad (9)$$

and, for all  $k \geq 1$

$$\begin{aligned} T_w x^k &= \frac{1}{w(k+1)} \sum_{h=0}^k \binom{k+1}{h} w^{k+1-h} x^h \\ &= \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} w^{k-h} x^h \\ &= \sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} w^{k-h} x^h + x^k \end{aligned} \quad (10)$$

Therefore the matrix of  $T_w$  in the basis  $\{1, \dots, x^N\}$  of  $\mathbb{R}_N[x]$ , we continue to denote it by  $T_w$ , has the form  $I + T(w)$  where  $T(w)$  is a lower triangular nilpotent matrix whose coefficients, with respect to the basis  $\{1, x, x^2, \dots, x^N\}$  of  $\mathbb{R}_N[x]$ , are given by

$$t_{mn}(w) = \chi_{\{m < n\}} \frac{n!}{(n+1-m)!m!} w^{n-m}, \quad 0 \leq m, n \leq N \quad (11)$$

Hence  $T_w$  is invertible because  $\det(T_w) = 1$ . Moreover, one has

$$T_w^{-1} = (I + T(w))^{-1} = \sum_{k=0}^N (-1)^k (T(w))^k \quad (12)$$

□

Let  $S$  denote the action of  $\mathbb{R}$  on  $\mathbb{R}_N[x]$  by translations

$$(S_u P')(x) := P'(x + u), \quad x, u \in \mathbb{R}$$

Then

$$T_u P' = \frac{1}{u} (S_u - id) P$$

where  $P$  is a primitive of  $P'$ .

The shift  $S_u$  is a linear homogeneous map of  $\mathbb{R}_N[x]$  into itself. Moreover

$$\sum_{k=0}^N a_k (x + u)^k = \sum_{k=0}^N \sum_{h=0}^k a_k \binom{k}{h} x^h u^{k-h}$$

$$\begin{aligned}
&= \sum_{h=0}^N \left( \sum_{k=h}^N a_k \binom{k}{h} u^{k-h} \right) x^h \\
&= \sum_{h=0}^N (S_u P')_h x^h
\end{aligned}$$

where

$$(S_u P')_h = \sum_{k=h}^N a_k \binom{k}{h} u^{k-h}$$

Thus, in the monomial basis  $\{1, \dots, x^N\}$  of  $\mathbb{R}_N[x]$ , the matrix associated to  $S_u$  is

$$(S_u)_{hk} = \chi_{\{h \leq k\}} \binom{k}{h} u^{k-h} \quad (13)$$

Note that  $S_u$  is invertible with inverse  $S_{-u}$  and

$$S_u S_v = S_{u+v}$$

Now denote  $\partial$  the action of derivation operator on  $\mathbb{R}_N[x]$ , i.e

$$\partial : \sum_{k=0}^N a_k x^k \rightarrow \sum_{k=1}^N k a_k x^{k-1} \quad (14)$$

Then clearly

$$\partial S_u = S_u \partial, \quad \forall u \in \mathbb{R} \quad (15)$$

Let  $\mathbb{R}_N[x]_0$  the subalgebra of  $\mathbb{R}_N[x]$  of all polynomials vanishing in zero

$$\mathbb{R}_N[x]_0 := \{P' \in \mathbb{R}_N[x] : P'(0) = 0\}$$

It is clear that

$$\mathbb{R}_N[x] = \mathbb{R}_N[x]_0 \oplus \mathbb{R}1$$

so that, thanks to (9),  $T_w$  is uniquely determined by its restriction to  $\mathbb{R}_N[x]_0$ .

Denote

$$\begin{aligned}
T_w^0 &:= T_w|_{\mathbb{R}_N[x]_0} \\
\partial_0 &:= \partial|_{\mathbb{R}_N[x]_0}
\end{aligned}$$

Then (14) implies that  $\partial_0$  is invertible and (15) implies that

$$\partial_0^{-1} S_u = S_u \partial_0^{-1}, \quad \forall u \in \mathbb{R} \quad (16)$$

**Lemma 2** *The following identities hold:*

- 1)  $T_w S_u = S_u T_w, \forall u, w \in \mathbb{R}$
- 2)  $T_w T_u = T_u T_w, \forall u, w \in \mathbb{R}$
- 3)  $T_u S_v = (1 + \frac{v}{u})T_{u+v} - \frac{v}{u}T_v, \forall u, v \in \mathbb{R} \text{ such that } u \neq 0$
- 4)  $T_u S_{-u} = T_{-u}, \forall u \in \mathbb{R}$
- 5)  $T_u T_v = (\frac{1}{v} + \frac{1}{u})T_{u+v} - \frac{1}{u}T_v - \frac{1}{v}T_u, \forall u, v \in \mathbb{R} \setminus \{0\}$

**Proof.** In the above notations

$$T_w^0 = \frac{1}{u}(S_u - id)\partial_0^{-1}$$

and (16) implies that

$$T_w^0 S_u = S_u T_w^0, \forall u, w \in \mathbb{R}$$

Since the constant function 1 is fixed both for  $S_u$  and for  $T_w$ , it follows that

$$T_w S_u = S_u T_w, \forall u, w \in \mathbb{R}$$

and therefore also

$$T_w T_u = T_u T_w, \forall u, w \in \mathbb{R}$$

which implies that assertions 1) and 2) are satisfied.

Now notice that translations commute with the operation of taking primitives, i.e. if  $P$  is a primitive of  $P'$ , then  $\forall v \in \mathbb{R}$ ,  $S_v P$  is a primitive of  $S_v P'$ , that is

$$\begin{aligned} T_u S_v P' &= \frac{1}{u}(S_u - id)S_v P \\ &= \frac{1}{u}(S_{u+v} - S_v)P \\ &= \frac{1}{u}(S_{u+v} - id)P + \frac{1}{u}(id - S_v)P \\ &= \left(\frac{u+v}{u}T_{u+v} - \frac{v}{u}T_v\right)P' \end{aligned}$$

This proves 3). In particular, choosing  $v = -u$ , one finds  $T_u S_{-u} = T_{-u}$  and 4) holds. Finally, from the identity

$$T_u T_v = \frac{1}{v}T_u S_v - \frac{1}{v}T_u$$

and from 3), the assertion 5) follows. □



## 2.2 Polynomial extensions of the Heisenberg algebra

Recall that the 1-dimensional Heisenberg group, denoted  $Heis(1)$ , is the nilpotent Lie group whose underlying manifold is  $\mathbb{R} \times \mathbb{R}^2$  and whose group law is given by

$$(t, z) \circ (t', z') = (t+t'-2\sigma(z, z'), z+z'); \quad t, t' \in \mathbb{R}; \quad z := (\alpha, \beta), z' := (\alpha', \beta') \in \mathbb{R}^2$$

where  $\sigma(\cdot, \cdot)$  the symplectic form on  $\mathbb{R}^2$  given by

$$\sigma(z, z') := \alpha\beta' - \alpha'\beta; \quad z := (\alpha, \beta), z' := (\alpha', \beta') \in \mathbb{R}^2$$

In the Schroedinger representation of the real Lie algebra  $Heis(1)$  one can define the *centrally extended Weyl operators* through the map

$$(t, z) \in \mathbb{R} \times \mathbb{R}^2 \mapsto W(t, z) := e^{i(\sqrt{2}(\beta q - \alpha p) + t1)} = W(z)e^{it} \quad (17)$$

where the Weyl operators are defined by

$$W(z) := e^{i\sqrt{2}(\beta q - \alpha p)} \quad ; \quad z := (\alpha, \beta) \in \mathbb{R}^2$$

From the Weyl relations

$$W(u)W(z) = e^{-i\sigma(u, z)}W(z+u) \quad (18)$$

one then deduces that

$$W(t, z)W(t', z') = W((t, z) \circ (t', z')) \quad (19)$$

i.e. that the map (19) gives a unitary representation of the group  $Heis(1)$ . The generators of the centrally extended Weyl operators have the form

$$up + a_1q + a_11 =: up + P'(q) \quad (20)$$

where  $u \in \mathbb{R}$  and  $P$  is a polynomial in one indeterminate with real coefficients of degree at most 1. Thus they are exactly the elements of the one dimensional Heisenberg algebra  $heis(1)$ .

Replacing  $P'$ , in (20), by a generic polynomial of degree at most  $N$  (in one indeterminate with real coefficients), one still obtains a real Lie algebra because of the identity

$$[up + P'(q), vp + Q'(q)] = u[p, Q'(q)] - v[p, P'(q)] = -iuQ''(q) + ivP''(q)$$

and, since  $\mathbb{R}_N[x]$  has dimension  $N+1$ , this real Lie algebra has dimension  $N+2$  on  $\mathbb{R}$ .

**Definition 1** For  $N \in \mathbb{N}$ , the real Lie algebra

$$\text{heis}(1, 1, N) := \{up + P'(q) \quad : \quad u \in \mathbb{R}, P' \in \mathbb{R}_N[x]\}$$

is called the  $(1, N)$ -polynomial extensions of the 1-dimensional Heisenberg algebra.

The notation  $(1, N)$  emphasizes that the polynomials involved have degree 1 in the momentum operator and degree  $\leq N$  in the position operator. With these notations the  $(1, N)$ -polynomial extensions of the centrally extended Weyl operators are the operators of the form

$$W(u, P') := e^{i(up + P'(q))} \quad ; \quad u \in \mathbb{R} \quad ; \quad P' \in \mathbb{R}_N[x] \quad (21)$$

By analogy with the 1-dimensional Heisenberg group, we expect that the pairs  $(u, P') \in \mathbb{R} \times \mathbb{R}_N[x]$  form a group for an appropriately defined composition law. The following theorem shows that this is indeed the case.

**Theorem 1** For any  $(u, P'), (v, Q') \in \mathbb{R} \times \mathbb{R}_N[x]$  one has

$$W(u, P')W(v, Q') = W((u, P') \circ (v, Q')) \quad (22)$$

where

$$(u, P') \circ (v, Q') := (u + v, T_{u+v}^{-1}(T_u P' + T_v S_u Q')) \quad (23)$$

**Proof.** From Proposition 1, one has

$$W(u, P') = e^{iup + iP'(q)} = e^{iT_u P'(q)} e^{iup}$$

Therefore

$$\begin{aligned} W(u, P')W(v, Q') &= e^{iup + iP'(q)} e^{ivp + iQ'(q)} \\ &= e^{iT_u P'(q)} e^{iup} e^{iT_v Q'(q)} e^{ivp} \\ &= e^{iT_u P'(q)} e^{iT_v Q'(q+u)} e^{i(u+v)p} \\ &= e^{i[T_u P'(q) + T_v Q'(q+u)]} e^{i(u+v)p} \\ &= e^{i(u+v)p + iT_{u+v}^{-1}(T_u P'(q) + T_v S_u Q'(q))} \\ &= W(u + v, T_{u+v}^{-1}(T_u P' + T_v S_u Q')) \end{aligned}$$

This proves (22) and (23). □

**Remark 1** *The associativity of the group law (23) can be directly verified using Lemma 2.*

Our next goal is to determine the  $(1, N)$ -polynomial extensions of the Weyl commutation relations. To this goal define the ideal

$$\mathbb{R}_N[x]_0 := \{P_0 \in \mathbb{R}_N[x] : P_0(0) = 0\} \quad (24)$$

Thus the projection map from  $\mathbb{R}_N[x]$  onto  $\mathbb{R}_N[x]_0$  is given by

$$P'(x) \in \mathbb{R}_N[x] \mapsto (P')_0(x) := P'(x) - P'(0) \in \mathbb{R}_N[x]_0 \quad (25)$$

and, in the notation (25), the  $(1, N)$ -polynomial extensions of the centrally extended Weyl operators (21) take the form

$$W(u, P') = e^{i(up+P'(q))} = e^{i(up+(P')_0(q))} e^{iP'(0)} \quad ; \quad u \in \mathbb{R} \quad ; \quad P' \in \mathbb{R}_N[x] \quad (26)$$

The analogy between (26) and (19) naturally suggests the following definition.

**Definition 2** *The unitary operators*

$$W(u, P'_0) := e^{i(up+P'_0(q))} \quad ; \quad u \in \mathbb{R} \quad ; \quad P'_0 \in \mathbb{R}_N[x]_0 \quad (27)$$

*will be called  $(1, N)$ -polynomially extended Weyl operators.*

From (22) and (23) we see that, if  $P'_0, Q'_0 \in \mathbb{R}_N[x]_0$ , then in the notation (25) one has

$$\begin{aligned} W(u, P'_0)W(v, Q'_0) &= W(u + v, T_{u+v}^{-1}(T_u P'_0 + T_v S_u Q'_0)) = \\ &= W(u + v, (T_{u+v}^{-1}(T_u P'_0 + T_v S_u Q'_0))_0) e^{i(T_{u+v}^{-1}(T_u P'_0 + T_v S_u Q'_0))(0)} \end{aligned} \quad (28)$$

By analogy with the usual Weyl relations we introduce the notation

$$\sigma((u, P'_0), (v, Q'_0)) := T_{u+v}^{-1}(T_u P'_0 + T_v S_u Q'_0)(0) \quad (29)$$

i.e.  $\sigma((u, P'_0), (v, Q'_0))$  is the degree zero coefficient of the polynomial  $T_{u+v}^{-1}(T_u P'_0 + T_v S_u Q'_0) \in \mathbb{R}_N[x]$ . Notice that the map

$$((u, P'_0), (v, Q'_0)) \equiv ((u, v), (Q'_0, P'_0)) \mapsto \sigma((u, P'_0), (v, Q'_0))$$

is linear in the pair  $(Q'_0, P'_0)$  but polynomial in the pair  $(u, v)$ . This is an effect of the duality between  $p$ , which appears at the first power, and  $q$ , which appears in polynomial expressions.

In order to prove the  $(1, N)$ -polynomial analogue of the classical Weyl relations (18) one has to compute the scalar factor (34). We will compute more generally all the coefficients of  $T_{u+v}^{-1}(T_u P'_1 + T_v S_u P'_2)$ . In view of (12) this leads to compute the powers of the matrices  $T(w)$  given by (8).

**Lemma 3** *Let  $k \in \{2, \dots, N\}$  and  $w \in \mathbb{R} \setminus \{0\}$ . Then, the matrix of  $(T(w))^k$ , in the monomial basis  $\{1, x, \dots, x^N\}$  of  $\mathbb{R}_N[x]$ , is given by  $(T_{ij}^{[k]}(w))_{0 \leq i, j \leq N}$ , where*

$$T_{ij}^{[k]}(w) = t_{ij}(w) C_{i,j}^{[k]} \quad (30)$$

the coefficients  $t_{ij}(w)$  are given by (8) and the  $C_{i,j}^{[k]}$  are inductively defined by

$$C_{i,j}^{[k]} = \sum_{h=0}^N C_{i,j}^{(h)} C_{i,h}^{[k-1]} \quad (31)$$

where

$$C_{i,j}^{(h)} = \frac{(j+1-i)!}{(h+1-i)!(j+1-h)!} \chi_{\{i < h < j\}}(h)$$

and  $C_{i,j}^{[1]} = \chi_{\{i < j\}}$  for all  $i, j = 0, \dots, N$ .

**Proof.** We prove the lemma by induction. For  $k = 2$ , one has

$$\begin{aligned} t_{ih}(w) t_{hj}(w) &= \chi_{\{i < h\}} \chi_{\{h < j\}} \frac{1}{h+1} \binom{h+1}{i} w^{h-i} \frac{1}{j+1} \binom{j+1}{h} w^{j-h} \\ &= \left[ \chi_{\{i < h < j\}} \frac{j!}{(j+1-i)! i!} w^{j-i} \right] \frac{(j+1-i)!}{(h+1-i)!(j+1-h)!} \\ &= t_{ij}(w) C_{i,j}^{(h)} \end{aligned} \quad (32)$$

where

$$C_{i,j}^{(h)} = \frac{(j+1-i)!}{(h+1-i)!(j+1-h)!} \chi_{\{i < h < j\}}$$

It follows that

$$T_{ij}^{[2]}(w) = t_{ij}(w) \sum_{h=0}^N C_{i,j}^{(h)} = t_{ij}(w) C_{i,j}^{[2]}$$

Now, let  $2 \leq k \leq N - 1$  and suppose that (30) holds true. Then, one has

$$T_{ij}^{[k+1]}(w) = \sum_{h=0}^N T_{ih}^{[k]}(w) t_{hj}(w)$$

Using induction assumption, one gets

$$T_{ij}^{[k+1]}(w) = \sum_{h=0}^N t_{ih}(w) t_{hj}(w) C_{i,h}^{[k]} \quad (33)$$

Therefore the identities (31), (32) and (33) imply that

$$T_{ij}^{[k+1]}(w) = t_{ij}(w) \sum_{h=0}^N C_{i,j}^{(h)} C_{i,h}^{[k]} = t_{ij}(w) C_{i,j}^{[k+1]}$$

□

As a consequence of Theorem 1 and Lemma 3, we are able to explicitly compute the scalar factor in the  $(1, N)$ -polynomial extensions of the Weyl relations (28).

**Proposition 2** *In the notation (34), for any  $u, v \in \mathbb{R}$  and for any  $P'_1, P'_2 \in \mathbb{R}_N[x]_0$  given by*

$$P'_1(x) = \alpha_1 x + \dots + \alpha_N x^N, \quad P'_2(x) = \beta_1 x + \dots + \beta_N x^N$$

*one has*

$$\begin{aligned} \sigma((u, P'_0), (v, Q'_0)) &= \sum_{j=0}^N \left[ \left( \frac{1}{j+1} (u+v)^j \sum_{m=0}^N (-1)^m C_{0,j}^{[m]} \right) \right. \\ &\quad \left. \left( \sum_{j \leq h \leq N} \left\{ t_{jh}(u) \alpha_h + t_{jh}(v) \sum_{h \leq k \leq N} (S_u)_{hk} \beta_k \right\} \right) \right] \end{aligned} \quad (34)$$

*where the coefficients  $(S_u)_{hk}$  are given by (13) and with the convention*

$$C_{0,j}^{[0]} = \delta_{j,0}, \quad \alpha_0 = \beta_0 = 0, \quad t_{jj}(w) = 1, \quad \text{for all } j = 0, \dots, N.$$

**Proof.** Let  $u, v \in \mathbb{R}$  and  $P'_1, P'_2 \in \mathbb{R}_N[x]_0$  be as in the statement. Then, using together Lemmas 1, 3 and identity (13), we show that

$$\begin{aligned} T_{u+v}^{-1}(T_u P'_1 + T_v S_u P'_2)(x) &= (T_{u+v}^{-1}(T_u P'_1 + T_v S_u P'_2))_0(x) \\ &+ \sum_{j=0}^N \left[ \left( \frac{1}{j+1} (u+v)^j \sum_{m=0}^N (-1)^m C_{0,j}^{[m]} \right) \right. \\ &\quad \left. \left( \sum_{j \leq h \leq N} \left\{ t_{jh}(u) \alpha_h + t_{jh}(v) \sum_{h \leq k \leq N} (S_u)_{hk} \beta_k \right\} \right) \right] \end{aligned}$$

□

**Definition 3** For  $N \in \mathbb{N}$ , the real Lie group with manifold

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}_N[x]_0 \equiv \mathbb{R} \times \mathbb{R}_N[x] \quad (35)$$

and with composition law given by

$$(u, P') \circ (v, Q') := (u + v, T_{u+v}^{-1}(T_u P' + T_v S_u Q'))$$

is called the  $(1, N)$ -polynomial extensions of the 1-dimensional Heisenberg group and denoted  $Heis(1, 1, N)$ .

**Remark 2** The left hand side of (35) emphasises that the coordinates  $(t, u, a_1, \dots, a_N) \in \mathbb{R}^{N+2}$  of an element of  $Heis(1, 1, N)$  are intuitively interpreted as time  $t$ , momentum  $u$  and coordinates of the first  $N$  powers of position  $(a_1, \dots, a_N)$ . Putting  $t = a_0$  is equivalent to the identification  $\mathbb{R} \times \mathbb{R}_N[x]_0 \equiv \mathbb{R}_N[x]$ .

### 2.3 The discrete Heisenberg algebra $\{1, p, q\}$

From Definition 1, it follows that  $\{1, p, q\}$  is a set of generators of  $heis(1, 1, 1)$  so that

$$heis(1, 1, 1) \equiv heis(1)$$

Moreover the composition law associated to  $Heis(1)$  is given by the following.

**Proposition 3** Let  $P'_1(x) = \alpha_0 + \alpha_1 x$ ,  $P'_2(x) = \beta_0 + \beta_1 x$  and let  $u, v \in \mathbb{R}$ . Then, we have

$$(u, P'_1) \circ (v, P'_2) = (u + v, Q')$$

where

$$Q'(x) = (\alpha_0 + \beta_0 + \frac{1}{2}u\beta_1 - \frac{1}{2}v\alpha_1) + (\alpha_1 + \beta_1)x$$

**Proof.** Let  $u, w \in \mathbb{R}$ . Then, one has

$$T_w = \begin{pmatrix} 1 & \frac{1}{2}w \\ 0 & 1 \end{pmatrix}, \quad T_w^{-1} = \begin{pmatrix} 1 & -\frac{1}{2}w \\ 0 & 1 \end{pmatrix} \quad S_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

Let  $P'_1(x) = \alpha_0 + \alpha_1x$ ,  $P'_2(x) = \beta_0 + \beta_1x$ . Then, from Theorem 1, the composition law is given by

$$(u, P'_1) \circ (v, P'_2) = (u + v, Q')$$

where

$$\begin{aligned} Q'(x) &= T_{u+v}^{-1}(T_u P'_1 + T_v S_u P'_2)(x) \\ &= (\alpha_0 + \beta_0 + \frac{1}{2}u\beta_1 - \frac{1}{2}v\alpha_1) + (\alpha_1 + \beta_1)x \end{aligned}$$

□

In particular recall that for all  $z = \alpha + i\beta$ , the Weyl operator  $W(z)$  is defined by

$$W(z) = W(\alpha, \beta) := e^{i\sqrt{2}(\beta q - \alpha p)}$$

Put

$$P'_1(x) = \beta\sqrt{2}x, \quad P'_2(x) = \beta'\sqrt{2}x$$

Then from the above proposition, one has

$$(-\sqrt{2}\alpha, P'_1) \circ (-\sqrt{2}\alpha', P'_2) = (-\sqrt{2}(\alpha + \alpha'), Q')$$

where

$$Q'(x) = (\alpha'\beta - \alpha\beta') + (\beta + \beta')\sqrt{2}x$$

Therefore, for all complex numbers  $z = \alpha + i\beta$ ,  $z' = \alpha' + i\beta'$ , it follows that

$$\begin{aligned} W(z)W(z') &= W(\alpha, \beta)W(\alpha', \beta') = e^{i(\alpha'\beta - \alpha\beta')} e^{i(\sqrt{2}(\beta + \beta')p - \sqrt{2}(\alpha + \alpha')p)} \\ &= e^{-i\Im(\bar{z}z')} W(z + z') \end{aligned}$$

**Remark 3** *The above proposition proves that the composition law given in Definition 3 is indeed a generalization of the composition law of the Heisenberg group.*

## 2.4 The discrete Galilei algebra $\{1, p, q, q^2\}$

In the case of  $N = 2$ , the composition law is given by the following.

**Proposition 4** *Let  $P'_1, P'_2 \in \mathbb{R}_2[x]$ , such that  $P'_1(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$  and  $P'_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ . Then, for all  $u, v \in \mathbb{R}$ , we have*

$$(u, P'_1) \circ (v, P'_2) = (u + v, Q')$$

where  $Q'(x) = \gamma + \beta x + \alpha x^2$  with

$$\begin{aligned} \gamma &= \alpha_0 + \beta_0 - \frac{1}{2}v\alpha_1 + \frac{1}{2}u\beta_1 + \frac{1}{6}(v - u)v\alpha_2 + \frac{1}{6}(u - v)u\beta_2 \\ \beta &= \alpha_1 + \beta_1 - v\alpha_2 + u\beta_2 \\ \alpha &= \alpha_2 + \beta_2 \end{aligned}$$

**Proof.** From Lemma 1, the matrices of  $T_w$  and  $T_w^{-1}$ , in the monomial basis  $\{1, x, x^2\}$  of  $\mathbb{R}_2[x]$ , are given by

$$T_w = \begin{pmatrix} 1 & \frac{1}{2}w & \frac{1}{3}w^2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \quad T_w^{-1} = \begin{pmatrix} 1 & -\frac{1}{2}w & \frac{1}{6}w^2 \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{pmatrix}$$

Moreover, identity (13) implies that

$$S_u = \begin{pmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{pmatrix}$$

Let  $P'_1(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ ,  $P'_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2$  and  $u, v \in \mathbb{R}$ . Then, using the explicit form of  $T_{u+v}^{-1}$ ,  $T_u$  and  $S_u$ , it is straightforward to show that

$$T_{u+v}^{-1}(T_u P'_1 + T_v S_u P'_2)(x) = \gamma + \beta x + \alpha x^2$$

where  $\alpha, \beta, \gamma$  are given in the above proposition. Finally, from Theorem 1 we can conclude.  $\square$

## 3 Polynomial extensions of the Heisenberg algebra: Continuous case

Let  $b_t^+$  and  $b_t$  satisfy the Boson commutation and duality relations

$$[b_t, b_s^+] = \delta(t - s) ; [b_t^+, b_s^+] = [b_t, b_s] = 0 ; (b_s)^* = b_s^+$$



where  $t, s \in \mathbb{R}$  and  $\delta$  is the Dirac delta function. Define

$$q_t = \frac{b_t + b_t^+}{\sqrt{2}} ; p_t = \frac{b_t - b_t^+}{i\sqrt{2}}$$

Then

$$[q_t, p_s] = i \delta(t - s) ; [q_t^k, p_s] = ik q_t^{k-1} \delta(t - s), \quad k \geq 1$$

$$[q_t, q_s] = [p_t, p_s] = [q_t^k, q_s^k] = [q_t^k, q_s] = 0$$

and

$$(q_s)^* = q_s ; (q_s^k)^* = q_s^k ; (p_s)^* = p_s$$

Now for all real function  $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we define

$$p(f) = \int_{\mathbb{R}} f(t) p_t dt, \quad q^k(f) = \int_{\mathbb{R}} f(t) q_t^k dt, \quad k \in \mathbb{N}$$

**Proposition 5** For all real functions  $g, f_1, \dots, f_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have

$$e^{i(p(g)+q^n(f_n)+\dots+q(f_1))} = e^{i \sum_{k=1}^n \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q^h(g^{k-h} f_k)} e^{ip(g)}$$

**Proof.** Consider real functions  $g, f_1, \dots, f_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Put

$$U_t = e^{it(p(g)+q^n(f_n)+\dots+q(f_1))} e^{-itp(g)}$$

Then one has

$$\begin{aligned} \partial_t U_t &= ie^{it(p(g)+q^n(f_n)+\dots+q(f_1))} (p(g) + q^n(f_n) + \dots + q(f_1)) e^{-itp(g)} \\ &\quad - ie^{it(p(g)+q^n(f_n)+\dots+q(f_1))} p(g) e^{-itp(g)} \\ &= ie^{it(p(g)+q^n(f_n)+\dots+q(f_1))} (q^n(f_n) + \dots + q(f_1)) e^{-itp(g)} \\ &= ie^{it(p(g)+q^n(f_n)+\dots+q(f_1))} e^{-itp(g)} [e^{itp(g)} (q^n(f_n) + \dots + q(f_1)) e^{-itp(g)}] \\ &= iU_t [e^{itp(g)} (q^n(f_n) + \dots + q(f_1)) e^{-itp(g)}] \end{aligned} \tag{36}$$

Note that

$$e^{itp(g)} q^k(f) e^{-itp(g)} = \int_{\mathbb{R}} e^{itp(g)} q_s^k e^{-itp(g)} f(s) ds$$

Put

$$V_{t,s} = e^{itp(g)} q_s^k e^{-itp(g)}$$

Then, one has

$$\begin{aligned} \partial_t V_{t,s} &= ie^{itp(g)} [p(g), q_s] e^{-itp(g)} \\ &= ie^{itp(g)} \int_{\mathbb{R}} g(u) [p_u, q_s] e^{-itp(g)} \\ &= \int_{\mathbb{R}} g(u) \delta(s - u) du = g(s) \end{aligned}$$

This gives

$$V_{t,s} = q_s + tg(s)$$

It follows that

$$e^{itp(g)} q_s^k e^{-itp(g)} = (q_s + tg(s))^k$$

Therefore, one gets

$$\begin{aligned} e^{itp(g)} q^k(f) e^{-itp(g)} &= \int_{\mathbb{R}} e^{itp(g)} q_s^k e^{-itp(g)} f(s) ds \\ &= \int_{\mathbb{R}} (q_s + tg(s))^k f(s) ds \\ &= \sum_{h=0}^k \binom{k}{h} t^{k-h} \int_{\mathbb{R}} q_s^h (g(s))^{k-h} f(s) ds \\ &= \sum_{h=0}^k \binom{k}{h} t^{k-h} q^h(g^{k-h} f) \end{aligned} \tag{37}$$

Using identities (36) and (37), one gets

$$\partial_t U_t = U_t \left( \sum_{k=1}^n \sum_{h=0}^k \binom{k}{h} t^{k-h} q^h(g^{k-h} f_k) \right)$$

This implies that

$$U_t = e^{i \sum_{k=1}^n \sum_{h=0}^k \frac{k! t^{k-h+1}}{(k+1-h)! h!} q^h(g^{k-h} f_k)}$$

Finally, one gets

$$e^{it(p(g)+q^n(f_n)+\dots+q(f_1))} = e^{i \sum_{k=1}^n \sum_{h=0}^k \frac{k! t^{k-h+1}}{(k+1-h)! h!} q^h(g^{k-h} f_k)} e^{itp(g)}$$

□

As a consequence of the above proposition, we prove the following.

**Lemma 4** *For all real functions  $g, f_1, \dots, f_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have*

$$e^{i(p(g)+q(f_1)+\dots+q^n(f_n))} = e^{iT_g(q(f_1)+\dots+q^n(f_n))} e^{ip(g)}$$

where

$$T_g q^k(f_k) = \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q^h(g^{k-h} f_k) = \int_{\mathbb{R}} T_{g(s)}(q_s^k) f_k(s) ds \quad (38)$$

with  $T_{g(s)} = I + T(g(s))$ , where, for all  $s \in \mathbb{R}$ ,  $T(g(s))$  is given by Lemma 1.

**Proof.** From Proposition 5, one has

$$\begin{aligned} e^{i(p(g)+q(f_1)+\dots+q^n(f_n))} &= e^{i \sum_{k=1}^n \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q^h(g^{k-h} f_k)} e^{ip(g)} \\ &= e^{i \sum_{k=1}^n T_g q^k(f_k)} e^{ip(g)} \end{aligned}$$

Moreover, one has

$$T_g q^k(f_k) = \int_{\mathbb{R}} \left( \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q_s^h g^{k-h}(s) \right) f_k(s) ds$$

Now using identity (10), it follows that

$$\sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q_s^h g^{k-h}(s) = T_{g(s)}(q_s^k)$$

This ends the proof. □

**Lemma 5** *For all real functions  $g, f_k \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have*

$$T_g^{-1} q^k(f_k) = \int_{\mathbb{R}} T_{g(s)}^{-1}(q_s^k) f_k(s) ds$$

**Proof.** Let  $g, f_k \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  be real functions. Define  $F_g$  as follows

$$F_g q^k(f_k) = \int_{\mathbb{R}} T_{g(s)}^{-1}(q_s^k) f_k(s) ds$$

Then one has

$$\begin{aligned}
F_g(T_g q^k(f_k)) &= \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} F_g q^h(g^{k-h} f_k) \\
&= \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} \int_{\mathbb{R}} T_{g(s)}^{-1}(q_s^h) g^{k-h}(s) f_k(s) ds \\
&= \int_{\mathbb{R}} T_{g(s)}^{-1} \left( \sum_{h=0}^k \frac{k!}{(k+1-h)!h!} q_s^h g^{k-h}(s) \right) f_k(s) ds \\
&= \int_{\mathbb{R}} (T_{g(s)}^{-1} T_{g(s)}(q_s^k)) f_k(s) ds \\
&= q^k(f_k) = T_g(F_g q^k(f_k))
\end{aligned}$$

which proves the required result.  $\square$

**Theorem 2** *Let  $g, f_1, \dots, f_n, G, F_1, \dots, F_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  be real functions. Then, the composition law associated to the continuous polynomial extensions of the Heisenberg algebra is given as follows*

$$\begin{aligned}
&(g, q(f_1) + \dots + q^n(f_n)) \circ (G, q(F_1) + \dots + q^n(F_n)) \\
&= (g + G, T_{g+G}^{-1}(T_g(q(f_1) + \dots + q^n(f_n)) + T_G S_g(q(F_1) + \dots + q^n(F_n))))
\end{aligned}$$

where for all real function  $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have

$$S_g q^k(f) = \int_{\mathbb{R}} (q_s + g(s))^k f(s) ds = \int_{\mathbb{R}} S_{g(s)}(q_s^k) f(s) ds \quad (39)$$

with  $S_{g(s)}$  is the translation operator.

**Proof.** Consider real functions  $g, f_1, \dots, f_n, G, F_1, \dots, F_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Then, from Lemma 4, one has

$$\begin{aligned}
&e^{i(p(g)+q(f_1)+\dots+q^n(f_n))} e^{i(p(G)+q(F_1)+\dots+q^n(F_n))} \\
&= e^{i(q(f_1)+\dots+q^n(f_n))} e^{ip(g)} e^{i(q(F_1)+\dots+q^n(F_n))} e^{ip(G)} \\
&= e^{i(q(f_1)+\dots+q^n(f_n))} \left( e^{ip(g)} e^{i(q(F_1)+\dots+q^n(F_n))} e^{-ip(g)} \right) e^{ip(g+G)} \\
&= e^{i(q(f_1)+\dots+q^n(f_n))} e^{i(e^{ip(g)} q(F_1) e^{-ip(g)} + \dots + e^{ip(g)} q^n(F_n) e^{-ip(g)})} e^{ip(g+G)}
\end{aligned}$$

But, from identity (37), one has

$$e^{ip(g)} q^k(F_k) e^{-ip(g)} = \int_{\mathbb{R}} (q_s + g(s))^k F_k(s) ds = S_g q^k(F_k)$$

This gives

$$\begin{aligned} & e^{i(p(g)+q(f_1)+\dots+q^n(f_n))} e^{i(p(G)+q(F_1)+\dots+q^n(F_n))} \\ &= e^{i(T_g(q(f_1)+\dots+q^n(f_n))+T_G S_g(q(F_1)+\dots+q^n(F_n)))} e^{ip(g+G)} \\ &= e^{i(p(g+G)+T_{g+G}^{-1}(T_g(q(f_1)+\dots+q^n(f_n))+T_G S_g(q(F_1)+\dots+q^n(F_n))))} \end{aligned}$$

which ends the proof of the above theorem.  $\square$

### 3.1 The continuous Heisenberg algebra $\{1, p(f), q(f)\}$

For all  $f \in L^2(\mathbb{R})$ , define

$$B(f) = \int_{\mathbb{R}} b_s \bar{f}(s) ds, \quad B^+(f) = \int_{\mathbb{R}} b_s^+ f(s) ds$$

Let  $f = f_1 + if_2$ ,  $g = g_1 + ig_2 \in L^2(\mathbb{R})$  such that  $f_i, g_i, i = 1, 2$ , are real functions. Then, a straightforward computation shows that

$$B(f) + B^+(f) = \sqrt{2}(q(f_1) + p(f_2)) = q(f'_1) + p(f'_2)$$

where for all  $i = 1, 2$ ,  $f'_i = \sqrt{2}f_i$ . Recall that the Weyl operator, associated to an element  $f = f_1 + if_2 \in L^2(\mathbb{R})$ , is defined by

$$W(f) = e^{i(B(f)+B^+(f))} = e^{i\sqrt{2}(q(f_1)+p(f_2))} = e^{i(q(f'_1)+p(f'_2))}$$

Note that from Theorem 2, one has

$$e^{i(q(f'_1)+p(f'_2))} e^{i(q(g'_1)+p(g'_2))} = e^{i(p(f'_2+g'_2)+T_{f'_2+g'_2}^{-1}(T_{f'_2} q(f'_1)+T_{g'_2} S_{f'_2} q(g'_1)))} \quad (40)$$

But, one has

$$T_{f'_2} q(f'_1) = \frac{1}{2} \int_{\mathbb{R}} f'_1(s) f'_2(s) ds + q(f'_1) \quad (41)$$

and

$$T_{g'_2} S_{f'_2} q(g'_1) = \int_{\mathbb{R}} g'_1(s) f'_2(s) ds + \frac{1}{2} \int_{\mathbb{R}} g'_1(s) g'_2(s) ds + q(g'_1) \quad (42)$$

Moreover, recall that the matrix of  $T_{f'_2(s)+g'_2(s)}^{-1}$  in the canonical basis  $\{1, x\}$  of  $\mathbb{R}_2[x]$  is given by

$$\begin{pmatrix} 1 & -\frac{1}{2}(f'_2(s) + g'_2(s)) \\ 0 & 1 \end{pmatrix}$$

This proves that

$$T_{f'_2+g'_2}^{-1}q(f'_1) = -\frac{1}{2} \int_{\mathbb{R}} f'_1(s)f'_2(s)ds - \frac{1}{2} \int_{\mathbb{R}} f'_1(s)g'_2(s)ds + q(f'_1) \quad (43)$$

$$T_{f'_2+g'_2}^{-1}q(g'_1) = -\frac{1}{2} \int_{\mathbb{R}} g'_1(s)f'_2(s)ds - \frac{1}{2} \int_{\mathbb{R}} g'_1(s)g'_2(s)ds + q(g'_1) \quad (44)$$

Therefore, using identities (40)–(44), one gets

$$e^{i(q(f'_1)+p(f'_2))}e^{i(q(g'_1)+p(g'_2))} = e^{\frac{i}{2}(\langle g'_1, f'_2 \rangle - \langle f'_1, g'_2 \rangle)}e^{i(q(f'_1+g'_1)+p(f'_2+g'_2))} \quad (45)$$

Finally, by taking  $f'_i = \sqrt{2}f_i$ ,  $g'_i = \sqrt{2}g_i$ ,  $i = 1, 2$ , in (45), one obtains the well known Weyl commutation relation

$$W(f)W(g) = e^{-i\Im\langle f, g \rangle}W(f + g)$$

### 3.2 The continuous Galilei algebra $\{1, p(f), q(f), q^2(f)\}$

The composition law associated to the continuous Galilei algebra  $\{1, p(f), q(f), q^2(f), f = \bar{f} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\}$  is giving by the following.

**Proposition 6** *For all real functions  $g, G, f_1, f_2, F_1, F_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have*

$$\begin{aligned} & \left( g, q(f_1) + q^2(f_2) \right) \circ \left( G, q(F_1) + q^2(F_2) \right) \\ &= \left( g + G, \frac{1}{2}\langle g, F_1 \rangle - \frac{1}{2}\langle G, f_1 \rangle + \frac{1}{6}\langle g, (g - G)F_2 \rangle + \frac{1}{6}\langle G, (G - g)f_2 \rangle \right. \\ & \quad \left. + q(f_1) + q(F_1) + q(gF_2) - q(Gf_2) + q^2(f_2) + q^2(F_2) \right) \end{aligned}$$

**Proof.** Recall that the matrices of  $T_{g(s)}$ ,  $S_{g(s)}$  and  $T_{g(s)+G(s)}^{-1}$  in the monomial basis  $\{1, x, x^2\}$  of  $\mathbb{R}_2[x]$  are given by

$$\begin{aligned} S_{g(s)} &= \begin{pmatrix} 1 & g(s) & g(s)^2 \\ 0 & 1 & 2g(s) \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{g(s)} = \begin{pmatrix} 1 & \frac{1}{2}g(s) & \frac{1}{3}g(s)^2 \\ 0 & 1 & g(s) \\ 0 & 0 & 1 \end{pmatrix}, \\ T_{g(s)+G(s)}^{-1} &= \begin{pmatrix} 1 & -\frac{1}{2}(g(s) + G(s)) & \frac{1}{6}(g(s) + G(s))^2 \\ 0 & 1 & -(g(s) + G(s)) \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Using all together the explicit form of the above matrices, identities (38), (39) and Lemma 5, one gets

$$\begin{aligned}
S_g(q(F_1) + q^2(F_2)) &= \langle g, F_1 \rangle + \langle g^2, F_2 \rangle + q(F_1) + 2q(gF_2) + q^2(F_2) \\
T_G S_g(q(F_1) + q^2(F_2)) &= \langle g, F_1 \rangle + \langle g^2, F_2 \rangle + \frac{1}{2} \langle G, F_1 \rangle + \langle G, gF_2 \rangle + \frac{1}{3} \langle G^2, F_2 \rangle \\
&\quad + q(F_1) + 2q(gF_2) + q(GF_2) + q^2(F_2) \\
T_g(q(f_1) + q^2(f_2)) &= \frac{1}{2} \langle g, f_1 \rangle + \frac{1}{3} \langle g^2, f_2 \rangle + q(f_1) + q(gf_2) + q^2(f_2) \\
T_{g+G}^{-1}(T_g(q(f_1) + q^2(f_2)) + T_G S_g(q(F_1) + q^2(F_2))) &= \frac{1}{2} \langle g, F_1 \rangle - \frac{1}{2} \langle G, f_1 \rangle \\
&\quad + \frac{1}{6} \langle g, (g - G)F_2 \rangle + \frac{1}{6} \langle G, (G - g)f_2 \rangle + q(f_1) + q(F_1) + q(gF_2) - q(Gf_2) \\
&\quad + q^2(f_2) + q^2(F_2)
\end{aligned}$$

Finally from Theorem 2 we can conclude. □

## 4 The q-projection method

Because of the relation (see Proposition 1)

$$e^{i\alpha(\sqrt{2}p)} e^{i\beta(\sqrt{2}q)} = e^{i2\alpha\beta} e^{i\beta(\sqrt{2}q)} e^{i\alpha(\sqrt{2}p)} \quad (46)$$

the Weyl algebra coincides with the complex linear span of the products  $e^{i\beta(\sqrt{2}q)} e^{i\alpha(\sqrt{2}p)}$ . Therefore a state on the Weyl algebra is completely determined by the expectation values of these products. In particular the Fock state  $\Phi$  is characterized by the property that the vacuum distribution of the position operator is the standard Gaussian

$$\sqrt{2}q \sim \mathcal{N}(0, 1)$$

together with the identity

$$e^{i\alpha(\sqrt{2}p)} \Phi = e^{-\alpha^2} e^{-\sqrt{2}\alpha q} \Phi \quad (47)$$

which follows from  $i\sqrt{2}\alpha p = -\sqrt{2}\alpha q + 2\alpha b$ .

The q-projection method consists in using (47) and (4) to reduce the problem to compute vacuum expectation values of products of the form  $e^{-izP(q)} e^{-iwp}$  to the calculation of a single Gaussian integral.

In the following sub-sections we illustrate this method starting from the simplest examples.

## 4.1 Vacuum characteristic functions of observables in $Heis(1, 1, 1)$

In this section we show that the q-projection method, applied to  $Heis(1)$ , gives the standard result for the spectral measure of the Weyl operators. From (47) and the CCR it follows that

$$e^{i(\alpha(\sqrt{2}p)+\beta(\sqrt{2}q))}\Phi = e^{i\alpha\beta}e^{-\alpha^2}e^{(i\beta-\alpha)(\sqrt{2}q)}\Phi$$

from which one obtains

$$\begin{aligned} \langle \Phi, e^{i(\alpha(\sqrt{2}p)+\beta(\sqrt{2}q))}\Phi \rangle &= e^{i\alpha\beta}e^{-\alpha^2} \langle \Phi, e^{(i\beta-\alpha)(\sqrt{2}q)}\Phi \rangle \\ &= \frac{e^{-\alpha^2}e^{i\alpha\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\beta x - \alpha x} e^{-\frac{x^2}{2}} dx \\ &= \frac{e^{-\frac{\alpha^2}{2}}e^{i\alpha\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\beta x} e^{-\frac{(x+\alpha)^2}{2}} dx \\ &= e^{-\frac{\alpha^2}{2}} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\beta x} e^{-\frac{x^2}{2}} dx \right] \\ &= e^{-\frac{\alpha^2}{2}} e^{-\frac{\beta^2}{2}} \end{aligned}$$

In particular, for all  $z = \alpha + i\beta$ ,  $z' = \alpha' + i\beta'$ , one has

$$\begin{aligned} \langle W(z)\Phi, W(z')\Phi \rangle &= \langle \Phi, W(-z)W(z')\Phi \rangle \\ &= e^{-i\Im(z\bar{z}')} \langle \Phi, e^{i\sqrt{2}((\beta'-\beta)q - (\alpha'-\alpha)p)}\Phi \rangle \\ &= e^{-i\Im(z\bar{z}')} e^{-\frac{\|z-z'\|^2}{2}} \end{aligned}$$

## 4.2 Vacuum characteristic functions of observables in $Heis(1, 1, 2)$

In this section we use the q-projection method to give a derivation of the expression of the vacuum characteristic functions of observables in  $\{1, p, q, q^2\}$  different from the one discussed in [x].

**Proposition 7** *For all,  $\alpha, \beta, \gamma \in \mathbb{R}$ , one has*

$$\langle \Phi, e^{i\alpha(\sqrt{2}q)^2 + i\beta(\sqrt{2}q) + \gamma(\sqrt{2}q)}\Phi \rangle = (1 - 2i\alpha)^{-\frac{1}{2}} e^{\frac{\gamma^2}{2(1-2i\alpha)}} e^{-\frac{\beta^2}{2(1-2i\alpha)}} e^{i\frac{\beta\gamma}{1-2i\alpha}}$$



**Proof.** Put

$$\Psi_1(\beta) := \langle \Phi, e^{i\alpha(\sqrt{2}q)^2 + i\beta(\sqrt{2}q) + \gamma(\sqrt{2}q)} \Phi \rangle = \mathbb{E}(e^{i\alpha X^2 + i\beta X + \gamma X})$$

where  $X$  is a normal gaussian random variable. Then, one has

$$\begin{aligned} \Psi_1'(\beta) &= i\mathbb{E}(X e^{i\alpha X^2 + i\beta X + \gamma X}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{i\alpha x^2 + i\beta x + \gamma x} e^{-\frac{x^2}{2}} dx \end{aligned} \quad (48)$$

Taking the changes of variables

$$u(x) = e^{i\alpha x^2 + i\beta x + \gamma x}, \quad v'(x) = x e^{-\frac{x^2}{2}}$$

Then, one gets

$$u'(x) = (2i\alpha x + i\beta + \gamma) e^{i\alpha x^2 + i\beta x + \gamma x}, \quad v(x) = -e^{-\frac{x^2}{2}}$$

This gives

$$\mathbb{E}(X e^{i\alpha X^2 + i\beta X + \gamma X}) = 2i\alpha \mathbb{E}(X e^{i\alpha X^2 + i\beta X + \gamma X}) + (i\beta + \gamma) \mathbb{E}(e^{i\alpha X^2 + i\beta X + \gamma X}) \quad (49)$$

Therefore, identities (48) and (49) imply that

$$\Psi_1'(\beta) = \frac{i\gamma - \beta}{1 - 2i\alpha} \Psi_1(\beta)$$

which yields

$$\Psi_1(\beta) = C(\alpha, \gamma) e^{i\frac{\beta\gamma}{1-2i\alpha}} e^{-\frac{\beta^2}{2(1-2i\alpha)}} \quad (50)$$

where

$$C(\alpha, \gamma) = \Psi_1(0) := \Psi_2(\gamma) = \mathbb{E}(e^{i\alpha X^2 + \gamma X})$$

Now, one has

$$\begin{aligned} \Psi_2'(\gamma) &= \mathbb{E}(X e^{i\alpha X^2 + \gamma X}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{i\alpha x^2 + \gamma x} e^{-\frac{x^2}{2}} dx \end{aligned}$$

Taking the changes of variables

$$h(x) = e^{i\alpha x^2 + \gamma x}, \quad l'(x) = x e^{-\frac{x^2}{2}}$$

It follows that

$$h'(x) = (2i\alpha x + \gamma)e^{i\alpha x^2 + \gamma x}, \quad l(x) = -e^{-\frac{x^2}{2}}$$

Then, one obtains

$$\Psi'_2(\gamma) = 2i\alpha\Psi'_2(\gamma) + \gamma\Psi_2(\gamma)$$

This implies that

$$\Psi_2(\gamma) = C(\alpha)e^{\frac{\gamma^2}{2(1-2i\alpha)}} \quad (51)$$

where

$$C(\alpha) = \Psi_2(0) = \mathbb{E}(e^{i\alpha X^2}) = (1 - 2i\alpha)^{-\frac{1}{2}} \quad (52)$$

Finally, using identities (50), (51) and (52), one obtains

$$\langle \Phi, e^{i\alpha(\sqrt{2}q)^2 + i\beta(\sqrt{2}q) + \gamma(\sqrt{2}q)} \Phi \rangle = (1 - 2i\alpha)^{-\frac{1}{2}} e^{\frac{\gamma^2}{2(1-2i\alpha)}} e^{-\frac{\beta^2}{2(1-2i\alpha)}} e^{i\frac{\beta\gamma}{1-2i\alpha}}$$

□

**Theorem 3** For all  $A, B, C \in \mathbb{R}$ , one has

$$\langle \Phi, e^{it(A(\sqrt{2}q)^2 + B(\sqrt{2}q) + C(\sqrt{2}p))} \Phi \rangle = (1 - 2itA)^{-\frac{1}{2}} e^{\frac{4C^2(A^2t^4 + 2iAt^3) - 3|M|^2t^2}{6(1-2iAt)}}$$

where  $M = B + iC$ .

**Proof.** We have

$$it(Aq^2 + Bq + Cp) = itCp + itP'(q)$$

where  $P(X) = \frac{1}{3}AX^3 + \frac{1}{2}BX^2$ . Then, Proposition 1 implies that

$$e^{it(Aq^2 + Bq + Cp)} = e^{it\frac{P(q+tC) - P(q)}{tC}} e^{itCp} \quad (53)$$

But, one has

$$\frac{P(q+tC) - P(q)}{tC} = Aq^2 + (tAC + B)q + \frac{1}{2}tBC + \frac{1}{3}A(tC)^2 \quad (54)$$

Using (53) and (54) for getting

$$e^{it(Aq^2 + Bq + Cp)} = e^{it\left(\frac{1}{3}A(tC)^2 + \frac{1}{2}tBC\right)} e^{it\left(Aq^2 + (tAC + B)q\right)} e^{itCp}$$

It follows that

$$e^{it(A(\sqrt{2}q)^2+B(\sqrt{2}q)+C(\sqrt{2}p))} = \frac{e^{it(\frac{4}{3}A(tC)^2+tBC)}e^{it(A(\sqrt{2}q)^2+(2tAC+B)(\sqrt{2}q))}}{e^{itC(\sqrt{2}p)}} \quad (55)$$

Therefore from (55), (47) and (47), one has

$$\begin{aligned} e^{it(A(\sqrt{2}q)^2+B(\sqrt{2}q)+C(\sqrt{2}p))}\Phi &= e^{it(\frac{4}{3}AC^2t^2+t(B+iC)C)}e^{it(A(\sqrt{2}q)^2+(2ACt+(B+iC))(\sqrt{2}q))}\Phi \\ &= e^{it(\frac{4}{3}AC^2t^2+tMC)}e^{it(A(\sqrt{2}q)^2+(2ACt+M)(\sqrt{2}q))}\Phi \end{aligned} \quad (56)$$

where  $M = B + iC$ . Now, by taking

$$\alpha = At, \quad \beta = 2ACt^2 + Bt, \quad \gamma = -Ct$$

and using Proposition 7, one gets

$$\begin{aligned} \langle \Phi, e^{it(A(\sqrt{2}q)^2+(2ACt+M)(\sqrt{2}q))}\Phi \rangle &= (1 - 2iAt)^{-\frac{1}{2}} e^{\frac{C^2t^2}{2(1-2iAt)}} e^{-\frac{(2At^2+Bt)^2}{2(1-2iAt)}} \\ &\quad e^{-i\frac{Ct(2ACt^2+Bt)}{1-2iAt}} \end{aligned} \quad (57)$$

Finally, identities (56) and (57) imply that

$$\langle \Phi, e^{it(A(\sqrt{2}q)^2+B(\sqrt{2}q)+C(\sqrt{2}p))}\Phi \rangle = (1 - 2itA)^{-\frac{1}{2}} e^{\frac{4C^2(A^2t^4+2iAt^3)-3|M|^2t^2}{6(1-2iAt)}}$$

This ends the proof.  $\square$

Using together Proposition 4 and Theorem 3, we prove the following theorem.

**Theorem 4** *For all  $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ ,  $i = 1, 2$ , we have*

$$\begin{aligned} &\langle e^{it(\alpha_1(\sqrt{2}q)^2+\beta_1(\sqrt{2}q)+\gamma_1(\sqrt{2}p))}\Phi, e^{it(\alpha_2(\sqrt{2}q)^2+\beta_2(\sqrt{2}q)+\gamma_2(\sqrt{2}p))}\Phi \rangle \\ &= (1 - 2i(\alpha_2 - \alpha_1)t)^{-\frac{1}{2}} e^{\frac{Lt^4+Z_1t^3+Z_2t^2}{6(1-2i(\alpha_2-\alpha_1)t)}} \end{aligned}$$

where

$$\begin{aligned} L &= -4(\alpha_1\gamma_2 - \alpha_2\gamma_1)[3(\alpha_1\gamma_2 - \alpha_2\gamma_1) + 2(\alpha_2 - \alpha_1)(\gamma_1 + \gamma_2)] \\ &\quad + 4(\gamma_2 - \gamma_1)^2(\alpha_2 - \alpha_1)^2 \\ Z_1 &= 12[(\alpha_2 - \alpha_1)(\beta_1\gamma_2 - \gamma_1\beta_2) - (\beta_2 - \beta_1)(\alpha_1\gamma_2 - \alpha_2\gamma_1)] \\ &\quad + 4i[2(\alpha_2 - \alpha_1)(\gamma_2 - \gamma_1)^2 + (\gamma_1 + \gamma_2)(\alpha_2\gamma_1 - \alpha_1\gamma_2)] \\ Z_2 &= -3(\gamma_2 - \gamma_1)^2 + 6i(\beta_1\gamma_2 - \gamma_1\beta_2) \end{aligned}$$

**Proof.** We have

$$\begin{aligned} & \langle e^{it(\alpha_1(\sqrt{2}q)^2 + \beta_1(\sqrt{2}q) + \gamma_1(\sqrt{2}p))} \Phi, e^{it(\alpha_2(\sqrt{2}q)^2 + \beta_2(\sqrt{2}q) + \gamma_2(\sqrt{2}p))} \Phi \rangle \\ &= \langle \Phi, e^{-it(\alpha_1(\sqrt{2}q)^2 + \beta_1(\sqrt{2}q) + \gamma_1(\sqrt{2}p))} e^{it(\alpha_2(\sqrt{2}q)^2 + \beta_2(\sqrt{2}q) + \gamma_2(\sqrt{2}p))} \Phi \rangle \end{aligned} \quad (58)$$

Put

$$\begin{aligned} P'_1(x) &= -t\sqrt{2}\beta_1x - 2t\alpha_1x^2 \\ P'_2(x) &= t\sqrt{2}\beta_2x + 2t\alpha_2x^2 \end{aligned}$$

Then, Proposition 4 implies that

$$(-t\sqrt{2}\gamma_1, P'_1)(t\sqrt{2}\gamma_2, P'_2) = (t\sqrt{2}(\gamma_2 - \gamma_1), Q') \quad (59)$$

where  $Q'(x) = C + Bx + Ax^2$  with

$$\begin{aligned} C &= (\beta_1\gamma_2 - \gamma_1\beta_2)t^2 + \frac{2}{3}(\gamma_1 + \gamma_2)(\alpha_2\gamma_1 - \alpha_1\gamma_2)t^3 \\ B &= \sqrt{2}(\beta_2 - \beta_1)t + 2\sqrt{2}(\alpha_1\gamma_2 - \alpha_2\gamma_1)t^2 \\ A &= 2(\alpha_2 - \alpha_1)t \end{aligned}$$

Therefore, from identities (58) and (59), one obtains

$$\begin{aligned} & \langle e^{it(\alpha_1(\sqrt{2}q)^2 + \beta_1(\sqrt{2}q) + \gamma_1(\sqrt{2}p))} \Phi, e^{it(\alpha_2(\sqrt{2}q)^2 + \beta_2(\sqrt{2}q) + \gamma_2(\sqrt{2}p))} \Phi \rangle \\ &= e^{i((\beta_1\gamma_2 - \gamma_1\beta_2)t^2 + \frac{2}{3}(\gamma_1 + \gamma_2)(\alpha_2\gamma_1 - \alpha_1\gamma_2)t^3)} \\ & \quad \langle \Phi, e^{it\left((\alpha_2 - \alpha_1)(\sqrt{2}q)^2 + ((\beta_2 - \beta_1) + 2(\alpha_1\gamma_2 - \alpha_2\gamma_1)t)(\sqrt{2}q) + (\gamma_2 - \gamma_1)(\sqrt{2}p)\right)} \Phi \rangle \end{aligned}$$

Now, from Theorem 3, one gets

$$\begin{aligned} & \langle e^{it(\alpha_1(\sqrt{2}q)^2 + \beta_1(\sqrt{2}q) + \gamma_1(\sqrt{2}p))} \Phi, e^{it(\alpha_2(\sqrt{2}q)^2 + \beta_2(\sqrt{2}q) + \gamma_2(\sqrt{2}p))} \Phi \rangle \\ &= e^{i((\beta_1\gamma_2 - \gamma_1\beta_2)t^2 + \frac{2}{3}(\gamma_1 + \gamma_2)(\alpha_2\gamma_1 - \alpha_1\gamma_2)t^3)} (1 - 2i(\alpha_2 - \alpha_1)t)^{-\frac{1}{2}} \\ & \quad e^{\frac{4(\gamma_2 - \gamma_1)^2((\alpha_2 - \alpha_1)^2t^4 + 2i(\alpha_2 - \alpha_1)t^3) - 3|M|^2t^2}{6(1 - 2i(\alpha_2 - \alpha_1)t)}} \end{aligned} \quad (60)$$

where

$$M = (\beta_2 - \beta_1) + 2(\alpha_1\gamma_2 - \alpha_2\gamma_1)t + i(\gamma_2 - \gamma_1)$$

Finally, using the identity (60), it is easy to show that

$$\begin{aligned} & \langle e^{it(\alpha_1(\sqrt{2}q)^2 + \beta_1(\sqrt{2}q) + \gamma_1(\sqrt{2}p))} \Phi, e^{it(\alpha_2(\sqrt{2}q)^2 + \beta_2(\sqrt{2}q) + \gamma_2(\sqrt{2}p))} \Phi \rangle \\ &= (1 - 2i(\alpha_2 - \alpha_1)t)^{-\frac{1}{2}} e^{\frac{Lt^4 + Z_1t^3 + Z_2t^2}{6(1 - 2i(\alpha_2 - \alpha_1)t)}} \end{aligned}$$

where  $L, Z_1, Z_2$  are given above.  $\square$

### 4.3 Vacuum moments of observables in $\{1, p, q, q^2\}$

In this subsection we use the results of the preceeding section to deduce the expression of the vacuum moments of observables in  $\{1, p, q, q^2\}$ . The form of these moments was not known and may be used to throw some light on the still open problem of finding the explicit expression of the probability distributions of these observables.

**Theorem 5** *Define, for  $A, B, C \in \mathbb{R}$*

$$X := A(\sqrt{2}q)^2 + B(\sqrt{2}q) + C(\sqrt{2}p)$$

*Then*

$$\langle \Phi, X^n \Phi \rangle = \sum_{i_1+2i_2+\dots+ki_k=n} \frac{2^{3n}n!}{i_1! \dots i_k!} w_1^{i_1} \dots w_k^{i_k}$$

*where*

$$w_k = \frac{A^k}{2k} - \frac{A^{k-2}}{4} \gamma \chi_{\{k \geq 2\}} - \frac{A^{k-3}}{8} \beta \chi_{\{k \geq 3\}} + \frac{A^{k-4}}{16} \alpha \chi_{\{k \geq 4\}}$$

*with*

$$\alpha = \frac{2}{3}A^2C^2, \quad \beta = \frac{4}{3}AC^2, \quad \gamma = -\frac{1}{2}|M|^2 = -\frac{1}{2}(B^2 + C^2)$$

**Proof.** Recall that

$$\mathbb{E}(e^{itX}) = e^{\varphi(t)}$$

where

$$\varphi(t) = -\frac{1}{2} \ln(1 - 2iAt) + \frac{\alpha t^4 + i\beta t^3 + \gamma t^2}{1 - 2iAt}$$

with  $\alpha, \beta$  and  $\gamma$  are given above. Hence, one has

$$\mathbb{E}(X^n) = \frac{1}{i^n} \left( \frac{d}{dt} \right)^n \Big|_{t=0} e^{\varphi(t)} \quad (61)$$

Now we introduce the following formula (cf [8])

$$\frac{d^n}{dt^n} e^{\varphi(t)} = \sum_{i_1+2i_2+\dots+ki_k=n} \frac{2^{2n}n!}{i_1! \dots i_k!} \left( \frac{\varphi^{(1)}(t)}{1!} \right)^{i_1} \dots \left( \frac{\varphi^{(k)}(t)}{k!} \right)^{i_k} e^{\varphi(t)} \quad (62)$$

Put

$$\begin{aligned} \varphi_1(t) &= \alpha t^4 + i\beta t^3 + \gamma t^2, \quad \varphi_2(t) = (1 - 2iAt)^{-1}, \\ \varphi_3(t) &= -\frac{1}{2} \ln(1 - 2iAt), \quad g(t) = \varphi_1(t)\varphi_2(t) \end{aligned}$$

Note that

$$\varphi_2^{(k)}(t) = (2iA)^k k! (1 - 2iAt)^{-k-1}, \varphi_3^{(k)}(t) = \frac{1}{2} (2iA)^k (k-1)! (1 - 2iAt)^{-k} \quad (63)$$

Then, one gets

$$g^{(k)}(t) = \sum_{h=0}^k \binom{k}{h} \varphi_1^{(h)}(t) \varphi_2^{(k-h)}(t)$$

Because  $\varphi_1^{(h)}(t) = 0$  for all  $h \geq 5$  and  $\varphi_1(0) = \varphi_1'(0) = 0$ , one gets

$$g^{(k)}(0) = \frac{k!}{(k-2)!} \gamma \varphi_2^{(k-2)}(0) + i \frac{k!}{(k-3)!} \beta \varphi_2^{(k-3)}(0) + \frac{k!}{(k-4)!} \alpha \varphi_2^{(k-4)}(0) \quad (64)$$

where by convention  $\varphi_2^{(i-j)}(0) = 0$  if  $i < j$ . Therefore, identities (63) and (64) imply that

$$g^{(k)}(0) = k! (2iA)^{k-2} \gamma \chi_{\{k \geq 2\}} + ik! (2iA)^{k-3} \beta \chi_{\{k \geq 3\}} + k! (2iA)^{k-4} \alpha \chi_{\{k \geq 4\}}$$

Then, for all  $k \geq 1$ , one obtains

$$\begin{aligned} \frac{\varphi^{(k)}(0)}{k!} &= \varphi_3^{(k)}(0) + g^{(k)}(0) \\ &= (2i)^k \left( \frac{A^k}{2k} - \frac{A^{k-2}}{4} \gamma \chi_{\{k \geq 2\}} - \frac{A^{k-3}}{8} \beta \chi_{\{k \geq 3\}} + \frac{A^{k-4}}{16} \alpha \chi_{\{k \geq 4\}} \right) \\ &= (2i)^k w_k \end{aligned}$$

where

$$w_k = \frac{A^k}{2k} - \frac{A^{k-2}}{4} \gamma \chi_{\{k \geq 2\}} - \frac{A^{k-3}}{8} \beta \chi_{\{k \geq 3\}} + \frac{A^{k-4}}{16} \alpha \chi_{\{k \geq 4\}}$$

Thus, from identity (62), one has

$$\begin{aligned} \left( \frac{d}{dt} \right)^n \Big|_{t=0} e^{\varphi(t)} &= \sum_{i_1+2i_2+\dots+ki_k=n} \frac{2^{2n} n!}{i_1! \dots i_k!} (2i)^{i_1+\dots+ki_k} w_1^{i_1} \dots w_k^{i_k} \\ &= i^n \sum_{i_1+2i_2+\dots+ki_k=n} \frac{2^{2n} 2^n n!}{i_1! \dots i_k!} w_1^{i_1} \dots w_k^{i_k} \end{aligned}$$

Finally, by using identity (61) the result of the above theorem holds true.  $\square$

#### 4.4 Vacuum characteristic functions of observables in $Heis(1, 1, N)$

From (2) and (47) we deduce that

$$\begin{aligned}\langle \Phi, e^{iwp + iuP'(q)} \Phi \rangle &= \langle \Phi, e^{iu \frac{P(q+w) - P(q)}{w}} e^{iwp} \Phi \rangle \\ &= \langle \Phi, e^{iu \frac{P(q+w) - P(q)}{w}} e^{iwp} \Phi \rangle \\ &= e^{-w^2/2} \langle \Phi, e^{iu \frac{P(q+w) - P(q)}{w}} e^{-wq} \Phi \rangle\end{aligned}$$

because

$$e^{i\alpha(\sqrt{2}p)} \Phi = e^{-\alpha^2} e^{-\sqrt{2}\alpha q} \Phi$$

The  $q$ -projection method reduces the problem to an integral of the form

$$\langle \Phi, e^{iQ(q)} \Phi \rangle$$

where  $Q$  is a polynomial.

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